4 Scale laws for linear-field FFAGs

4.1 Cells versus turns

The number of lattice cells, \( N_c \), and turns, \( N_t \), are important parameters for the machine and its operation. \( N_c \) and \( N_t \) are related by the constraint \( N_c N_t \langle \cos x \rangle = \Delta E/\delta E \), and cannot be chosen independently. Here \( \Delta E \) and \( \delta E \) are the machine energy range and per-cell energy increment, respectively. Typically \( \langle \cos x \rangle \) approaches unity.

The number of cells is obtained from the per cell phase-slip condition and per cell time-of-flight variation:

\[
\frac{\delta E}{\Delta E} \geq a \omega \Delta T, \quad \Delta T \approx \frac{3}{4} \frac{(\Delta p)^2 \theta^2}{(\frac{\Delta p}{\bar{p}})^2 (1 - \cos \Phi) T_0}.
\]

Here \( a(b) \) is a dimensionless acceptance parameter and \( \omega \) is the cavity angular frequency. \( \theta = \pi/N_c \) is the bend half-angle and \( T_0 \) is the cell time of flight at the reference momentum. Here \( \Phi \) is the per cell horizontal betatron phase advance at the injection momenta \( \bar{p} \). \( \Delta p \) and \( \delta p \) are the momentum range of the machine and the impulse per cell, respectively.

The cell length is contributed to by the cavity \( L_{\text{cav}} \) and magnets \( L_{\text{mag}} \); included are drift spaces. A rough estimate of the magnet lengths is \( L_{\text{mag}} \approx 3R\theta \) where \( R = p/(eB) \) is the D bend radius; \( B \) is the magnetic bending field, and \( e \) the charge. Hence the relation:

\[
\frac{\delta E}{\Delta E} \geq \frac{3a}{4(1 - \cos \Phi)} \left( \frac{\Delta p}{\bar{p}} \right)^2 \frac{\omega}{c} \left( L_{\text{cav}} + 3R\theta \right) \theta^2.
\]

This is a cubic equation to be solved for \( \theta \propto 1/N_c \). Two simplifying extremes, the cavity dominated and the magnet dominated, can be identified; the intermediate case is complicated. In the relativistic regime, \( \beta \to 1, \delta p/\Delta p \) may be substituted in place of \( \delta E/\Delta E \).

4.1.1 Cavity dominated regime

In the cavity and/or drift-space dominated regime, the magnets are considered relatively short, and so this is a many cell regime. The number of cells and turns are proportional to the following:

\[
N_c \propto \frac{\Delta p}{\bar{p}} \sqrt{\frac{a L_{\text{cav}} \omega/c}{(1 - \cos \Phi) \delta p}}, \quad N_t \propto \bar{p} \sqrt{\frac{(1 - \cos \Phi)}{a L_{\text{cav}} \omega/c} \frac{1}{\Delta p \delta p}}.
\]

4.1.2 Magnet dominated regime

In the magnet dominated regime, the magnets are comparatively long and there are few cells. The number of cells and turns are proportional to the following:

\[
N_c \propto \left( \frac{a R \omega/c}{(1 - \cos \Phi) \delta p} \frac{\bar{p}}{\delta p} \right)^{1/3}, \quad N_t \propto \left( \frac{(1 - \cos \Phi)}{a R \omega/c} \left( \frac{\bar{p}}{\delta p} \right)^2 \right)^{1/3}.
\]
In all cases, a large number of turns (and small number of cells) is favoured by small acceptance, low frequency, short cells (short cavity/drift and/or small bend radius) and large betatron phase advance; typically $\Phi \approx 180^\circ$ is adopted. Further, many turns favours an injection momentum $\tilde{p}$ which is large in comparison to the cell increment $\delta p$. In the muon FFAG designs, $\delta p$ is held constant for a cascade of three machines with differing injection momenta, and so the low energy ring\(^1\) makes fewer turns than the high energy ring. In the case of a large momentum range, the number of cells grows and the machine looks increasingly like a linac. Typically the ratio of momenta $\Delta p/\tilde{p}$ is taken to be unity.

4.2 Beyond PoP for the electron model

Typical designs for a 10-20 GeV muon FFAG have a lattice with 90-100 cells, and a machine circumference of roughly 450 m. Early on in the electron model design, decisions were taken to reduce the scale to an office-sized device having $\approx 30$ cells and 15 m circumference. The motivation was to make the model as small and inexpensive as possible, so long as consistent with demonstrating “proof of principle” (PoP). Clearly the lengths of magnets and cavity had to be reduced, and the bend angles increased; $\approx 20$ cm total magnet length (per cell) and 3 GHz frequency were selected. The following article seeks to understand what refinements are required to go beyond PoP, what does it take to make a scale model?

4.3 RF scaling

For acceleration in the serpentine-shape rotation manifold, there is the fundamental phase-slip condition $\delta p/\Delta p \geq a(\omega \Delta T)$. Ideally, this demands large impulse $\delta p$, low radio-frequency, and small ToF variation. The rf system design has some control over $\delta p$ and $\omega$. The former is limited by electric field gradient and breakdown; and both are constrained by the availability of proven rf structure designs. For the case of multi-GeV muons, an energy gain of $\approx 10$ MeV/cell and frequency of 200 MHz has been selected for the superconducting cavities. Adequate longitudinal acceptance an acceleration is achieved with $a = \frac{1}{12}$ in the 10-20 GeV ring. In the lower energy rings, 2.5-5 and 5-10 GeV, larger acceptances $a = \frac{1}{6}, \frac{1}{8}$ are required and so it is almost inevitable that the FFAG cascade operate with similar $\delta E$ per cell.

For the case of multi-MeV electrons, field breakdown is not an issue because the energy gain per turn is roughly 1000 times smaller; in principle, a single cavity with $\approx 1$ MeV gap voltage would surface. However, for reasons of maintaining structural similarity between the machines, reducing synchro-betatron coupling, and prohibiting the motion from becoming too discretized, the ratio $\delta p/\Delta p$ will be similar between the muon and electron machines; that is, for the latter, much lower than technically feasible. This has three consequences: (i) a gap voltage of 10-100 keV, depending on the number of cells; (ii) more accelerating cavities in the electron model than would be recommended by cost minimization; and (iii) the product $a \omega \Delta T$ must take a value comparable with the muon FFAGs. The choice of $\omega$ and $\Delta T$ are both impacted by the over-riding consideration that the model should be small and low cost - which favours few and short cells.

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\(^1\)The lower energy rings have also fewer cells and weaker bending fields.
Varying the acceptance parameter \( a \) is part of the longitudinal dynamics investigation in the electron model. Because there is no strong limitation of the electric field, values spanning \( a = [0, \frac{1}{4}] \) are considered. For the basis of the lattice design \( a = \frac{1}{12} \) is assumed; the machine will always work with larger \( a \) values and smaller number of turns.

### 4.4 Lattice and component scaling

For strong similarity between the muon and electron machines, the ratios \( \delta p / \Delta p \) and products \( a \omega \Delta T \) should be equal between them as should be the betatron phase advances.

From equations (1), we wish to scale \( \omega T_0 \), and \( \theta \), between the muon and electron machines - the relative momentum ranges and betatron tunes being kept equal. The cell length contributions \( L_{\text{cav}} \) and \( L_{\text{mag}} \) include flanges, bellows and coil over-hang etc. In most of the lattice designs, the per cell F-element length is roughly \( \frac{1}{3} - \frac{1}{2} \) of the D-element. A rough estimates of the cavity length is \( L_{\text{cav}} \approx (3/4) \lambda_{\text{rf}} \) with \( \lambda \) the free-space wavelength. Hence the relation:

\[
\omega T_0 = \frac{\omega}{c} (L_{\text{cav}} + L_{\text{mag}}) \approx \frac{3\pi}{2} + \frac{\omega}{c} 3R \theta \, .
\]  

(5)

Because the product of frequency and wavelength is always \( c \), an important conclusion is that the cavity contribution to the per-cell rf phase advance is independent of \( \omega \); and will be equal, automatically, between model and muon FFAG. However, the magnet (and drift) contribution is scaled by the radio-frequency. Although it is far from a rigid constraint, a more exact correspondence between the muon and electron FFAGs suggests that the ratios \( L_{\text{cav}} / L_{\text{mag}} \) be the same between the machines.

If the magnet field and bend angles were kept the same between the muon and electron machines, the magnet lengths would be given by the ratio of momenta, 1000, leading to mm-size magnets in the electron model. This is contrary to the requirement of cm-sized components for ease of fabrication and by-hand assembly. Hence the magnet field is relaxed from 5 tesla (for muons) to 0.2 tesla. (The various electron model lattices have \( L_{\text{mag}} \) varying from 15-20 cm, whereas the muon lattice have \( L_{\text{mag}} \) varying between 1-2 m, depending on lattice type and number of cells.) The magnet length has been scaled down by a factor of \( \frac{25}{1000} = \frac{1}{40} \). Thus, assuming equal bend angle, the frequency could be allowed to rise by a factor up to 40; from 200 MHz to 8 GHz. However, there are three arguments against this: (i) the aperture is too small for the radial sweep; (ii) the small cavity size (< 3 cm) compromises our principle of easy by-hand assembly; and (iii) there is no suitable structure at this frequency. For those reasons a lower frequency, 3 GHz, was chosen due to the wide availability of components. At this point, the electron model may look less challenging than its muon counterpart; but looks can be deceiving.

It must not be forgotten that the model has (up to this point) one third the number of cells, and three times the bend angle per cell of the muon FFAG. Thus, while the \( L_{\text{mag}} \) contribution to the per-cell phase advance \( \omega T_0 \) is almost equal \( (3 \times \frac{3}{8} \approx 1) \) between the electron and muon FFAGs, the per-cell phase slip due to time-of-flight variation, \( \omega \Delta T \propto \omega T_0 \theta^2 \), could be nine times larger. There are a variety of ways out of this predicament: (i) increase the number of cells; (ii) reduce the radio-frequency; (iii) increase the per-cell energy increment; and (iv) slight shrinkage of components and their spacings. All methods may be
used and combined in varying degrees.

Given the relation $N_c N_t \langle \cos x \rangle = \Delta E/\delta E$, a third of the cells, and the stipulation of equal turns $N_t$, it is natural to increase the per-cell impulse by a factor three. Combined with a frequency reduction to 1.3 GHz, this is enough to re-establish the inequality $\delta E/\Delta E \geq a\omega\Delta T$.

Alternatively, the bend angle may be reduced; due to the quadratic/cubic variation advantage is swiftly accumulated. A 50% increase in number of cells, from 30 to 45, leads to a 2.5 reduction in $\omega\Delta T$. Combined with a factor 2.4 increase in $\delta E/\Delta E$, c.f. muons, the phase slip condition is satisfied.

Actually, the number of turns $N_t$ is not a variable to be freely stipulated, and so the final optimization is more complex, but qualitatively the arguments are correct. Other things being equal, the optimum solution will be that using the minimum number of cells.

**FURTHER DISCUSSION OF NUMBER OF CAVITIES IS NEEDED?**

Could one, for instance, release some cells for diagnostics, etc? and compensate by raising voltage elsewhere? Answer is probably in the affirmative, because rf voltage envisaged for electron model is far from physical limits.